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Foundations of Topological Order:

Quantum Topological States of Matter with Applications in Solid State Physics, Quantum Computing, and Quantum Information Theory

A White Paper on Applications of

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See Also:

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1. Classification of Global Topological and Geometric Quantum Phases

The notion of a global topological or geometric phase was introduced in quantum mechanics by Sir M. Berry in 1984 [1, 12]. Indeed, all typical global quantum mechanical observables are measured in terms relative phases obtained by interference phenomena. These phenomena involve various splitting and recombination processes of beams whose global coherence is measured precisely by some relative phase difference. A relative phase can be thought of as the physical attribute measuring the global coherence between two histories of events sharing a common initial and final point in the space of some control variables of the dynamical evolution of a quantum system, parameterized implicitly by some temporal parameter. For example, we may think of the simplest case of a beam which is split into two beams propagating for a period of time and finally recombined. Their interference is always measured by a global relative phase difference.

As mentioned above, it was Sir M. Berry who first formulated a general mechanism depicting the generation of an experimentally observed global phase factor of a geometric or topological origin. It has since been shown that a quantum system undergoing a slowly evolving (adiabatic) cyclic evolution retains a “memory” of its motion after coming back to its original physical state. This “memory” is expressed by means of a complex phase factor in the wave-function (state vector) of the system, typically referred to as the “Berry phase” or simply the geometric phase. The cyclic evolution, which can be thought of as a periodicity property of the state vector of a quantum system, is driven by a Hamiltonian bearing an implicit time dependence through a set of control variables. For instance, we may think of external electric or magnetic fields which define the Hamiltonian parametric dependence of a charged particle. The adiabatic condition defines a constraint of parallel transport specified by the requirement that the implicit time dependence of the Hamiltonian is sufficiently slow so that the state vector stays in the eigenspace of the same instantaneous eigenvalue of the Hamiltonian.

Intuitively, once the state vector is prepared in an instantaneous eigenstate of the Hamiltonian with an eigenvalue which is separated from the neighboring eigenstates by a finite energy gap, then it remains there during its transport within a finite temporal period. We may think of the space of control variables as a slowly varying environment with respect to which a state vector (eigenvector of the Hamiltonian localized at the corresponding eigenspace) displays a history dependent geometric effect: When the environment returns to its original state, the system also does, *but for an additional global geometric phase factor*. **Due to the implicit temporal dependence imposed by the time parameterization of a closed path in the environmental parameters of the control space, this global geometric phase factor is thought of as “memory” of the motion since it encodes the global geometric or topological features of the control space.**

The Berry phase is a complex number of modulus one and is experimentally observable. The two most important features regarding the experimental detection of a quantum global phase are [i] that it is a statistical object, and [ii] it can be measured only relatively. Thus it becomes observable by comparing the historical evolution of two distinct statistical ensembles of systems through their interference pattern. The Berry phase is geometric or topological because it depends solely on the topology or geometry of the control space pathway along which the state vector is transported. It does *not* depend on either the temporal metric duration of the evolution or on the particular dynamics that is applied to the system.

The important conceptual lesson to be learned from the experimental discovery of Berry-type phases is that although quantum mechanics may be locally interpreted in terms of probabilities of events (so that phases do not play any role), *globally it is the relative phase differences between histories of events that have the major physical significance*. **Failure to recognize this subtle point focusing on the distinctive role of the topologically local and global levels of quantum mechanical description in relation to physical observability and information has caused enormous interpretational problems. These problems, in turn, have become the central impediment to the development of viable technological applications of quantum mechanical phenomena.**

What has been lacking, until now, is a coherent, consistent theoretical framework which describes the internal topological relations between the local and the global level of quantum mechanical descriptions of phenomena, so that these are not ad hoc or disjoint. In our recent book *Foundations of Relational Realism: A Topological Approach to Quantum Mechanics and the Philosophy of Nature* [2] we have developed precisely such a scheme (see also [17, 18, 19]), which is formulated mathematically in the category-theoretic language of sheaf theory [3, 6, 7].

A sheaf is an algebraic-topological object depicting the integration of local structural information into an induced global structure over some base space of control variables—the topological 'gluing conditions' generative of a global topological covering structure of families of local reference frames. In general, a sheaf may be thought of as a *continuously variable relational information structure*, whose continuous variation is considered over specified local covering frames interlocking together non-trivially.

To date, our conceptual and technical scheme is the only one which specifically predicts the appearance of global topological phase factors without any additional ad hoc hypotheses (like the adiabatic hypothesis). This is due to the fact that the concept of a sheaf, on which our scheme is based, *explicitly incorporates the distinction between global and local information carriers as well as their concrete internal relations in contradistinction to all other interpretations of quantum phase phenomena*. The key modeling notion of a quantum information sheaf leads to the following principle central to the understanding of the origin of global phase factors: Whenever a global information totality is partitioned into local relational parts—i.e., whenever they are localized contextually with respect to particular information carriers (forming a covering system parameterizing this totality)—and one attempts to describe some part in isolation from an environment of other parts (e.g., in a conventional quantum measurement interaction, where a measured system is practically considered isolated from its environment), the connectivity between the global environment and local measured system (due to fact that they are topologically glued together in the same global totality) is manifested as global observable phase factors of a topological or geometric origin.

Our sheaf-theoretic framework uniquely provides a clear demarcation of the minimum physical requirements for the qualification and quantification of all global relative phase phenomena as follows: [i] The local gauge freedom of each localized part with respect to some symmetry condition (like the local gauge freedom of the phase of a state vector), which leads to its theoretical representation as a fiber or stalk of an information sheaf; [ii] The global constraints or obstructions imposed by the topological structure of the base space of control variables. These can be quantified by means of closed paths (loops) and their generalizations, which probe these global constraints by topologically bounding them—e.g., by encircling a hole or an impurity or an inaccessible region. All different types of loops can be classified in terms of homotopy-theoretic global invariants; [iii] The encoding of this global information in differential extensive terms so that it can be accessed inductively by using local information carriers and their interlocking properties—e.g., by solving partial differential equations; [iv] The non-trivial topological information of global significance is measured (after the differential encoding in terms of the local carriers) via an integration procedure of an associated differential gauge potential (technically referred to as a ‘connection’), which acts along a constraint-bounding contour (for example a closed path or loop) and which is implicitly parameterized by a continuous temporal parameter; [v] The gauge potentials incorporate the connectivity properties of information from the local to the global level in differential terms. The gauge character expresses the allowed local contextual variability or freedom of a potential with respect to some global topological constraint; [vi] The global invariant topological information is finally measured and expressed in terms of global (an)holonomy phase factors induced by integration of differential gauge potentials along constraint-bounding contours.

The next major theoretical challenge is the classification of *all* possible global topological or geometric phase factors. **This task is of crucial significance for the technological applications of global phase phenomena in condensed matter physics, electronics and quantum computing.** From the perspective of our theoretical scheme, which is based on the concepts and principles of differential sheaf theory according to the above brief exposition, the natural methodology is provided by the analytic technique of sheaf cohomology. For this reason, **we propose that sheaf cohomology is the appropriate technical device for the complete classification of all global phase phenomena, creating astonishing prospects of concrete technological applications backed up by a solid mathematical model instead of various phenomenological simulations involving unnecessary or even unphysical and unrealistic hypotheses.**

> The conception of this project is original and its primary innovation centers on the idea of applying the powerful methods of sheaf cohomology for the complete classification of global phase phenomena over topological spaces of control variables bearing highly-non-trivial topological information, which can be subsequently used for technological purposes. This project will thus contribute substantially to the creation of new knowledge related to technological applications of topological phases as an efficient mechanism for coding global information, which can be accessed and processed by local means.

Sheaf cohomology can be best thought of as a method of assigning global invariants to a topological space of control variables (or more generally to a categorical covering scheme called a site) in a homotopy-invariant way. The sheaf cohomology groups measure the global obstructions for extending information represented in terms of sheaf sections from the local to the global level (for example extending local solutions of a differential equation to a global solution).

2. Technological Application of Global Quantum Phases: Superconducting Topological Insulators

The particular significance of the concepts of relative topological and geometric phases in quantum mechanics from the viewpoint of our theoretical scheme is that they mark a distinctive point in the history of science, where *for the first time the significance of global observables or global information carriers as distinct entities from local observables is realized and made explicit through precise physical models—models which have found concrete experimental applications*.

The most exciting current applications of the discovery of global topological phase factors encoding the information of global topological invariants in the structure and emergence of various materials come from condensed matter and solid state physics.

> Due to the huge potential in the applicability of these new quantum states of matter in various technological fields, physics has seen a recent surge of experimental activity related to *topological states of matter* and *topological order*. These new materials, called *topological insulators,* can act as both insulators and conductors [4].

More precisely, while the interior of topological insulators prevents the flow of electrical currents, their edges or surfaces allow the movement of charges. Most important, the surfaces of topological insulators enable the transport of spin-polarized electrons while preventing any scattering effects resulting in dissipation. **Because of these characteristics, topological insulators hold great potential for use in future transistors, memory devices and magnetic sensors that are highly energy efficient and require less power.** It is instructive to think of these materials as crystals being able to conduct electrical current on their surfaces, while acting as insulators throughout the interior of the crystal. Thus, a topological insulator always has a metallic boundary when compared to an ordinary insulator. *These metallic boundaries originate from global topological invariants, which are not sensitive to any continuous small perturbations caused for instance from thermal fluctuations, as long as a material remains insulating*.

In our *Foundations of Relational Realism: A Topological Approach to Quantum Mechanics* [2], one of the major conclusions has been the theoretical prediction, justification and mathematical elaboration of the fact that **the understanding of quantum event structures and quantum information requires a conceptual shift in our thinking of the notion of extension from the *metrical order* to the *topological order***. It is the *topological* relations between local information carriers and the mutually-implicative role between local-global levels (captured concretely by the notion of a sheaf), as opposed to the *metrical* relations (on which the conventional quantum formalism is based), that are central to the coherent and consistent conceptualization and mathematical modeling of the quantum world.

It is exciting to realize that this theoretical perspective on the nature and functioning of the quantum world has now been experimentally confirmed, and that this confirmation has revolutionized the disciplines of condensed matter physics and solid state physics in a way that points to practical technological applications. The major purpose of research in condensed-matter physics is to explain from first principles how order in matter emerges when a large number of simple constituents, such as ions, or electrons, interact with each other. In metrically ordered phases such as those pertaining to crystals and magnets, the order is described via the mechanism of *symmetry breaking*: for instance the order in a crystal is obtained by the symmetry breaking of the Euclidean group of rotations and translations of the three-dimensional metrical space, because ions are arranged periodically due to their electrostatic interactions. Until the experimental discovery of the topological quantum Hall effect, it had been thought that all states of matter could be classified in terms of their broken symmetries giving rise to characteristic local metrical order parameters. *However, this classification fails for the quantum Hall state of matter which is classified by a global topological invariant* [18]. The quantum Hall effect occurs when a magnetic field penetrates a two-dimensional, low temperature conducting sample of electrons, and pertains to the quantization of the resulting conductivity of the sample. In this way, the quantum Hall state has been the first experimentally observed *topological state of matter*. Topological states with different values of the global topological invariant can agree in all symmetries. Thus, topologically distinct matter states cannot be adiabatically (viz. preserving the gap among energy levels) deformed continuously to each other as long as they share the same symmetries. Thus, the global topological invariant classifying these states of matter plays a protecting role being totally unaffected by perturbations and deformations. **Conclusively, there exists a topological type of order underlying the quantum Hall state of matter, which surpasses the old metrical paradigm.**

One of the most important discoveries of the past few years is that a similar type of topological order also occurs in some three-dimensional materials, *called topological insulators* or *quantum Hall spin states* [4, 5, 10]. In the case of topological insulators, the role of the magnetic field is assumed by the mechanism of spin–orbit coupling, which is an intrinsic property of all solids. In this topological state of matter in three dimensions *and even at room temperatures*, materials can *insulate on the inside* but *conduct on the outside*. Moreover, the conducting electrons on the surface arrange themselves into spin-up electrons traveling in one direction, and spin-down electrons travelling in the other, generating a “spin current,” which can have transport properties without dissipation. Thus, *the potential technological applications of these topological states of matter in electronics and quantum computing is enormous*. An interesting question is whether it would be possible to simulate a topological insulator using light, so that to obtain an optical analogue, called a *photonic topological insulator*. This has been positively answered already in a very recent work [11], where the first experimental realization of a photonic topological insulator has been reported, which consists of helical waveguides arranged in a honeycomb lattice.

The next challenge, which is expected to play a major and catalytic role in quantum computing and spin electronics is the theoretical investigation and practical implementation of hybrid organization quantum material structures. One of the most promising among them is the *superconducting topological insulator* [4, 5], which involves the *coupling of a superconductor with a topological insulator*.

More precisely, it is predicted that the appropriate coupling of the boundary surface of a topological insulator to a superconductor can induce a type of “superconducting spin current” transporting information *without* dissipation (i.e., without energy exchange). This superconducting information transport is expected to be carried by particular fermionic modes called “Majorana fermions” [14], which according to Dirac’s theoretical model would be their own antiparticles.

The proof of existence of “Majorana fermions” would be a tremendous scientific achievement with amazing technological applications; *for example they could play the role of quantum bits in a solid state implementation of quantum computing*. We may think of these fermionic modes as vortices on the surface with a memory of their topological localization relative to all other vortices, such that *it becomes theoretically possible to implement all quantum computing operations by controlling the relative topological localization properties of these vortices*. To this end, we propose to apply our sheaf-theoretic model of classification of global topological phases in order to gain a deep understanding of [a] the nature of coupling between a superconductor and a topological insulator, and [b] of the origin of the dissipationless surface superconducting current induced by this coupling.

According to our model, global topological invariants are encoded in terms of global topological phase factors (memories) in the states of the corresponding fermionic modes. Theoretically they are obtained if we consider the structure of a sheaf, which relationally localizes the fermion eigenstates with respect to some base space of control parameters, which depend implicitly on time. In solid state physics, the localizing space of control parameters is considered to be the momentum space (Brillouin zone), so that the energy spectrum has a band structure, meaning that it is piecewise continuous. **Thus, we expect that different bulk topological constrains of a topological insulator would lead to different types of “Majorana fermions” with respect to the kind of quantum memory they carry.** In this manner, the understanding and implementation of superconducting topological insulators requires a topological classification in three dimensions of the quantum memories of these fermionic modes in relation to the variability of the possible topological constraints of the localization space of control variables. The crucial point is that this classification is relevant to the coupling of a superconductor with a topological insulator if and only if it refers to interacting fermionic modes and not to the free case, which is currently addressed in the literature [4, 5].

For this purpose, we propose the application of sheaf cohomological methods in order to tackle this problem. This is a novel approach to the subject of superconducting topological insulators and interacting topological states of matter in general, since the powerful concepts and methods of sheaf theory have not been considered at all in this discipline. **At a further stage of development, this research directly addresses the problem of interactions and strong correlations pertaining to topological states of matter, which could potentially lead to a substantial revision of the Standard Model of field interactions.**

3. Applications of Global Topological Phases to Quantum Information Theory and Homotopic Quantum Computing

Quantum information theory and quantum computing must be an integral part of the world view of anyone who seeks a fundamental understanding of quantum theory and its implications regarding information processing and storing. Although quantum geometric spectrums may be locally probed in terms of observables, represented as self-adjoint operators, and their corresponding probabilities of events with respect to an orthonormal basis of eigenstates comprising a Boolean logical frame, so that local phases do not have any measurable significance, globally it is precisely the measurable relative phase differences which maintain the quantum coherence information. A global phase factor is not represented by any self-adjoint operator, but it is represented by means of an (an)holonomy unitary group element, to be thought of as the accumulated "memory" due to periodicity with respect to an environment of control variables. **The explicitly different nature of physical information carriers as we make the transition from the local to the global level of description of quantum geometric spectrums and inversely, requires an adequate quantum information processing scheme where this distinction is appropriately modeled.**

We propose that a natural approach to quantum information processing, which explicates precisely the fundamental difference between local and global information carriers, can be utilized by applying the theory of vector sheaves equipped with a connection [7, 8, 9].

Mathematically speaking, according to the Serre-Swan theorem [7], finitely generated projective modules, and thus locally free sheaves of modules, called vector sheaves, defined over commutative observable algebra sheaves, are equivalent to vector bundles over a paracompact and Hausdorff topological base space. Thus, the set of sections of any vector bundle encoding, for example, the physical information of quantum states always forms a vector sheaf. We emphasize that a base topological space of control variables serves only as the carrier of a bundle geometric spectrum, and in particular it incorporates the local/global distinction required for the sheaf-theoretic interpretation of this spectrum.

In view of the gauge-theoretic nature of quantum geometric spectrums the following aspects acquire particular significance: First, in the case of fiber bundle gauge geometry, the fiber over each point of a base space represents the local gauge freedom in the local definition of a physical information attribute. For instance, the vector space over each point of a base topological space of a vector bundle of quantum states represents the local gauge freedom in the local definition of a state. Therefore, from a computational viewpoint, each fiber serves as a local information encoding space. Second, due to equivalence of vector bundles with vector sheaves there should be **naturally utilized a sheaf-theoretic computational model of the bundle geometric spectrum, and in particular an information attribute at a point should be computed in terms of sheaf germs (compatible information equivalence classes) and not in a punctual way as it is classically the case**. Third, the crucial property of fiber bundle gauge geometric spectrums is that they bear the homotopy lifting property, meaning that homotopic loops on the base space (loops that can be continuously deformed to each other) can be lifted uniformly to the fibers of the bundle.

This is a clear indication that the proposed sheaf-theoretic quantum computational model should operate on homotopy types and not on set types. For this reason, **we propose to investigate a quantum information processing model in relation to homotopy type theory [19], which we call homotopic quantum computing**. The conception of this project is novel and the interpretation of quantum computation types as homotopy types provides a promising way to overcome the current problems of quantum computation based on set-theoretic types.

Homotopy type theory is a currently developing new field of mathematics and logic, which interprets type theory not from a set-theoretic, but from a homotopy-theoretic perspective [19]. In homotopy type theory one regards the types as spaces and the logical operations as homotopy-invariant constructions on spaces. In our case, we propose to consider quantum types as fiber bundle gauge geometric spectrums, such that a term of some specified quantum type is a germ of the associated vector sheaf. The key new idea of the homotopy interpretation in relation to quantum information processing is that the logical notion of equivalence of two terms of the same type (interpreted as a form of identity by the univalence axiom) is understood as follows: Two terms of the same type are equivalent if there exists a path of connectivity between them. From this perspective, quantum information processing can be expressed by means of a connection on the associated vector sheaf of a vector bundle quantum spectrum. Thus, an integrable connection on the vector sheaf of states of a quantum system provides a realization model of homotopic quantum computation.

The notion of a connection is formulated in sheaf-theoretic terms as a natural transformation from the vector sheaf of states (finite rank locally free sheaf of modules) over the observable algebra sheaf to the sheaf of vector-valued algebraic differential forms [7]. This natural transformation should obey the Leibniz rule, so that in effect can be interpreted physically as a covariant derivative of the sections of the vector sheaf of states. The variation takes place according to a differential parallel transport rule, which characterizes the specific manner of quantum information processing. In this way, the transport constraint on sections is expressed by means of a differential equation. The connection can be locally identified with a gauge potential according to the paradigm of gauge theories (for example electromagnetism). Thus, locally a gauge potential connects two infinitesimally close fibers along an infinitesimal path on the base space, and thus can be integrated along a finite extension of this path parameterized by a temporal parameter.

Due to global topological obstructions, the considered finite path extension cannot be covered by a single local chart on the base localization space. Hence, we need to consider a multitude of overlapping local charts covering the path extension forming a chain. Together with each local chart on the base space there is an associated local gauge potential representing the global connection with respect to this chart. Thus, we need to define compatibility conditions on pairwise overlaps gluing local gauge potentials together. This is expressed in terms of *transition functions*, which establish the transformation rules of the local potentials under extension from the local to the global level.

In this implicitly temporal model of a finite extent path on the base localization space, we may consider a closed path by looping the localizing parameters back to themselves during some finite time period. A closed path is homotopically non-trivial if it encloses a hole (for example an inaccessible region) so that it cannot be contracted continuously to a point.

Depending on the topological connectivity properties of the base space of parameters, closed paths can be classified homotopically by means of global invariants, for example the winding number of a loop tracing a circle. This homotopic information can be encoded in terms of a group, called the fundamental group of the base space, for example the group of integers for the case of the circle. In case that the base space is not simply connected, then there is always going to be some *global observable phase factor (unitary group element interpreted as a memory) of a topological origin, called the (an)holonomy of the connection.*

This topological phase factor is a global information carrier of the quantum information processing according to the utilized connection. More concretely, it is derived by integration in the temporal completion of the lifting procedure of a closed loop in the base space to the sections of the bundle geometric spectrum according to the connection. In particular, if the connection is integrable, then quantum information processing from the local to the global does not depend on the particular loop traced on the base space, but only on the homotopy equivalence class of such loops. **In this manner, an integrable connection on the vector sheaf of states of a quantum system can be thought of as a model of homotopic quantum computation**, under the proviso that quantum computation types are interpreted as homotopy types.

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